Particular attention is paid to the use of interdisciplinary connections, the application of mathematics to solve practical problems, the use of modern technologies, and the development of students' independent work skills.

The authors emphasize that the competency-based approach promotes not only the assimilation of theoretical knowledge but also the development of students' practical skills necessary for successful professional activity."

Key words: *Competency-based approach, algebra, mathematical analysis, senior specialized schools, critical thinking, problem-solving thinking, educational challenges, teaching methodology, mathematical competencies.*

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DEVELOPING SPATIAL IMAGINATION IN MATHEMATICS EDUCATION

Boykina D. Developing Spatial Imagination in Mathematics Education.

This article explores a range of effective approaches and strategies for fostering spatial imagination in students within the context of mathematics education, particularly from grades 5 through 12. Spatial imagination – the ability to mentally visualize and manipulate objects and their relationships in space – is a fundamental component of mathematical thinking and problem-solving. It plays a vital role not only in geometry, but also in algebra, trigonometry, and real-world applications such as engineering, architecture, and computer graphics.

The article emphasizes the importance of nurturing spatial reasoning early in students' academic development and presents a variety of methods to stimulate and strengthen this cognitive skill. These include illustrative examples, thought-provoking questions, and problem-based tasks designed to encourage visual thinking, abstract reasoning, and the ability to connect different mathematical concepts. Special attention is given to problems that involve working with three-dimensional figures, exploring spatial transformations, and interpreting complex visual information.

Key words: *spatial imagination, spatial thinking, spatial skills, problem solving.*

Problem statement. In modern pedagogical science, the development of spatial imagination is recognized as a key factor in acquiring mathematical knowledge and fostering abstract-logical thinking. Spatial imagination refers to the ability to mentally construct, transform, and manipulate images and objects in a mental space. This ability is directly linked to mastering geometric concepts, understanding algebraic structures, and solving logical-mathematical problems.

The formation of spatial thinking is not possible without the presence of imagination. Imagination is a cognitive process through which the real world is reflected in the human mind in the form of new, unusual, or even impossible images, ideas, or representations. This process involves various mental operations and activities, such as analysis, synthesis, and abstraction. It is well known that every learning activity engages a wide range of mental processes, including memorization, storage, reproduction of information, and of course, thinking.

Analysis of current research. The issue of imagination is not new. Psychologists have emphasized its importance for many years, yet it remains highly relevant today, as many students lack sufficiently developed spatial imagination. Studies by researchers such as Piaget [5], Vygotsky [7], and Bruner [1] highlight the significance of visual-spatial skills in the development of mathematical literacy. In the context of school education, these skills begin to develop in the early elementary years but become increasingly important in the middle and high school stages (grades 5–12), where learning involves complex spatial representations, graphical relationships, and analytical models.

The early formation and ongoing development of spatial imagination in both primary and secondary education contribute to greater efficiency in the learning process. However, this development may lose much of its effectiveness if it is not continuously nurtured and integrated into the learning process. Therefore, it must be deliberately stimulated, guided, and implemented systematically and purposefully.

According to Miller and Halpern [4], students with well-developed spatial skills tend to achieve higher outcomes in STEM disciplines, including mathematics. This finding indicates that instructional approaches aimed at fostering spatial imagination may play a crucial role in enhancing students' mathematical motivation and academic achievement.

It should be noted that various means can be used to form and develop spatial imagination, such as tasks (non-standard, engaging, etc.), practical exercises, and so on.

The objectives of the article is to propose effective methods and tools for stimulating spatial imagination in mathematics education for students in grades 5–12.

Findings. To ensure a systematic approach to the development of spatial imagination in students, we believe it is advisable to begin its formation even with younger learners, using tasks that do not explicitly mention the concept of space. For example, suitable tasks may be the following.

Example 1. Divide a round cake into eight pieces with only three cuts.

The answer is presented in Fig. 1. It shows the three cuts of the cake with dotted lines.

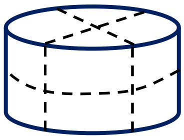

Fig. 1

Example 2. From six matchsticks, form four equilateral triangles so that the sides of each of them are a whole match.

The answer in this example is a triangular pyramid with an edge equal to one matchstick.

Example 3. With 12 sticks, build 6 squares [2, p. 11].

The answer here is a cube with an edge equal to one stick.

Example 4. With 12 sticks, build 4 squares. The answer here is shown on fig. 2, 3, 4.

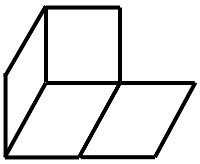

Fig. 2

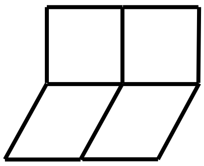

Fig. 3

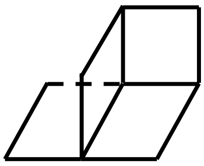

Fig. 4

Example 5. Four sticks make four right angles. Add one more stick to make eight right angles.

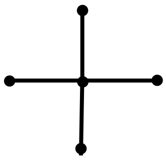

Fig. 5

Answer: It is appropriate to consider different possibilities here.

1. If the first four sticks are arranged in one plane and share a common endpoint O, the added fifth stick must be perpendicular to this plane and have a common endpoint with the other four sticks.
2. However, if three of the first four sticks lie in one plane and share a common endpoint O, while the fourth is perpendicular to this plane and passes through O, the added fifth stick must lie in the given plane and serve as an extension of the common stick forming the two right angles in that plane.

Note: In both configurations, a regular quadrilateral pyramid is essentially formed, with a height equal to half the diagonal of the base.

Another group of problems that can be used is related to discovering and "interpreting" what is depicted in a given figure (for example, Fig. 6 and 7). Any mathematician will recognize a cube in Fig. 6, rather than two squares with their vertices connected in pairs.

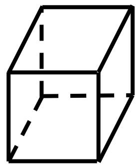

Fig. 6

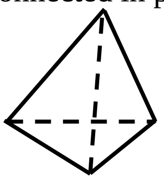

Fig. 7

In Fig. 7, the mathematician will see a triangular pyramid, not a quadrilateral with two diagonals.

The third group of problems consists of engaging tasks. Here, we will provide a few examples.

Example 6. How many faces does a hexagonal pencil have?

The answer is 8, if the pencil is not sharpened.

Example 7. A cube is made of paper (Fig. 8). It is easy to establish that its net can be cut into 6 equal squares – its faces. Is it possible to reconstruct this net into 12 equal squares?

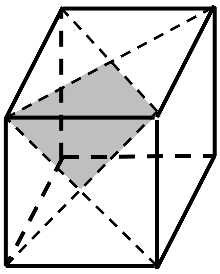

Fig. 8

It is not difficult to see here that the figure, formed by the union of triangles A and B, which are parts of the top base and the front face, respectively (Fig. 8), will be a square if they are placed in one plane.

Example 8. In how many triangles is the small square involved in Fig. 9?

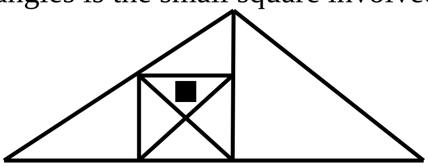

Fig. 9

The answer is: in six triangles.

Example 9. A farmer had a square-shaped yard. At the four corners of the yard, he planted one tree at each corner. After some time, he doubled the size of his yard, while maintaining its square shape. How did he manage to do this so that the trees remained at the fence? [3, p. 287].

The solution of the problem is shown on Fig. 10.

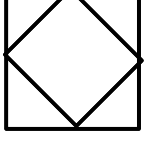

Fig. 10

If his yard was originally the square $ABCD$, then he built the square $MNPQ$ such that $MN \parallel AC$ and $NP \parallel BD$.

These problems not only contribute to the development of imagination but also spark interest among the learners, fostering motivation to acquire new knowledge. Generating interest is one of the necessary conditions for successfully studying the mathematical curriculum.

We will also note that the main tool for assisting in solving a problem is the drawing. It will differ depending on the observer's position relative to the same object. While the structure of the object (its shape and the relationships between its parts) remains unchanged, the projection of the object onto the plane will vary depending on which view is considered the main one. Along with that, the images of these projections will also change. It should be kept in mind that all of this is also connected to the realization of the principle of visual representation. Therefore, as we emphasized earlier, the main tool for assisting in solving problems is drawings. That is why it is necessary, even for younger students, to develop a system of tasks and questions that would support the formation of their imagination.

We present to the reader a sample system of such problems and questions.

Problem 1. List all the edges of the rectangular parallelepiped (Fig. 11) that are parallel to edge EF.

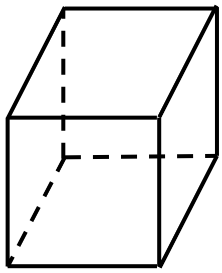

Fig.11

Problem 2. In Fig. 12, a rectangular parallelepiped is shown, and Fig. 13 displays its unfolded form (net). One of the faces of the parallelepiped is shaded in gray in both figures. Mark on Fig.12 the line segment AB from its net (Fig. 13).

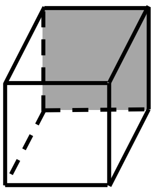

Fig. 12

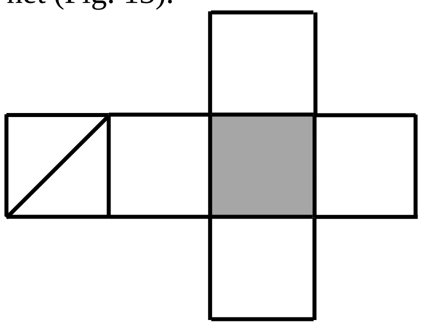

Fig. 13

Problem 3. In Fig. 14, a cube and its net are shown. Mark points A, B, and C, which are vertices of the cube, on the net.

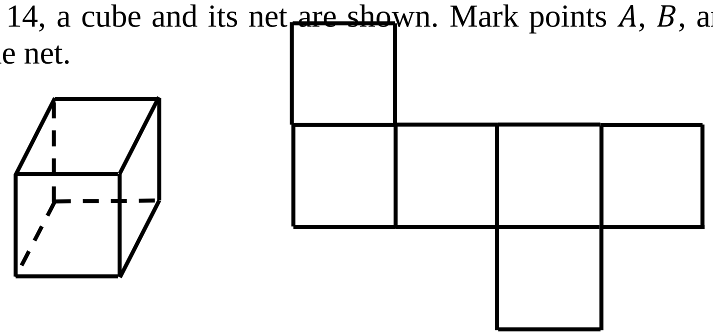

Fig. 14

Such problems can also be included in tests to assess the level of spatial imagination developed in students.

We offer a set of questions related to the topic "Prism" (studied in Grade 6).

1. In which of the Fig.15(a), 15(b), and 15(c) a prism is shown?

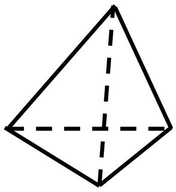

Fig. 15(a)

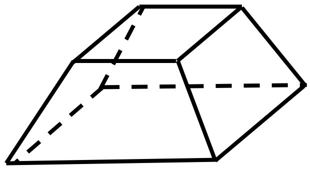

Fig. 15(b)

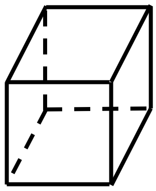

Fig. 15(c)

2. Indicate which of the following statements is NOT true:
a) Every prism has two bases;
b) In every right prism, the lateral faces are parallelograms;
c) In every right prism, any two non-parallel edges are perpendicular to each other.

3. From Fig.15(c), two arbitrary lines containing edges of a cube are selected. Which of the following statements about these lines is NOT true?
a) the lines are parallel;
b) the lines are perpendicular;
c) the lines intersect at an angle other than a right angle.

4. In which of the Fig. 16(a), 16(b), and 16(c) the letter h is NOT correctly placed for the given geometric body (if h represents the height of the body)?

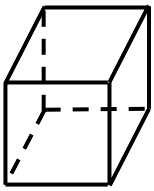

Fig. 16(a)

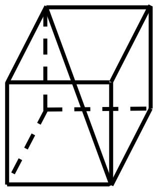

Fig. 16(b)

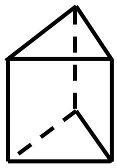

Fig. 16(c)

5. Which of the figures 17(a), 17(b), and 17(c) is NOT a net of a cube?

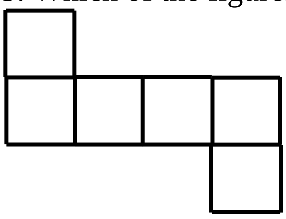

Fig. 17(a)

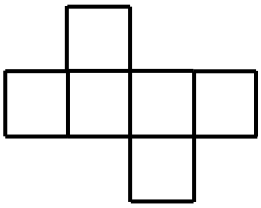

Fig. 17(b)

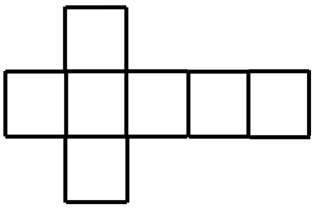

Fig. 17(c)

6. In Fig.18, a rectangular parallelepiped is shown, with some of its elements marked. How many of these elements are necessary to calculate the volume of this parallelepiped? Which of them are sufficient to determine its volume?

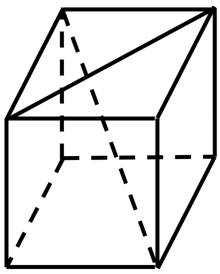

Fig. 18

7. Is it true that every rectangular parallelepiped has:
a) 12 edges and 6 faces;
b) 8 vertices and 12 edges?

In a similar way, a set of questions on the topic "Pyramid" for Grade 6 can be created. We will list only a few of the questions:

1. In Fig. 19, a pyramid is shown with a parallelogram as its base. Which of the following statements is true?

a) The lines AD and EF intersect at a common point;

b) The lines EK and BD are perpendicular;

c) The lines EF and AC are not parallel.

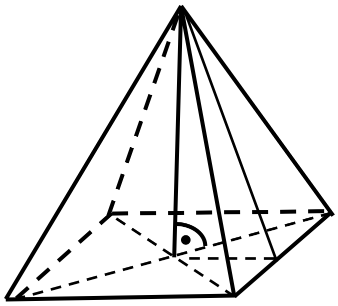

Fig. 19

2**.** Is it true that the net of a regular square pyramid consists of the following plane figures:

a) A rectangle and four equilateral triangles?

b) A square and four right-angled triangles?

c) A square and four equilateral triangles?

3. Which of the following statements is true?

a) A triangular pyramid has 4 vertices and 5 edges.

b) A triangular pyramid has 3 faces and 6 edges.

c) A triangular pyramid has 4 faces and 4 vertices.

Questions can also be asked on the topic "Cylindrical Solids" in Grade 6.

We will focus also on one of the types of geometric problems that pose a challenge for students, namely the section of a polyhedron by a plane. We will also discuss some of the methods for solving this type of problem. Each problem involving the section of a polyhedron by a given plane consists of two parts:

1. Constructing the section of the polyhedron by the given plane and determining the type of the section.

2. Performing the necessary calculations to find the answer to the problem (for example: finding the area of the section, the relationships between the edges of the polyhedron, or its volume, etc.).

In [6, p. 157-174], these questions are described in detail. The following problems will illustrate the idea of solving problems related to the sections of a polyhedron by a plane.

Problem 4. Construct the section of the pyramid $ABCS$ by the plane α, which passes through points D and E, lying on edges AS and BS, respectively. The points divide the edges in the ratio $SD:DA = S\text{E}:EB = 1:2$, and the plane is parallel to the edge SC.. Determine the type of the section.

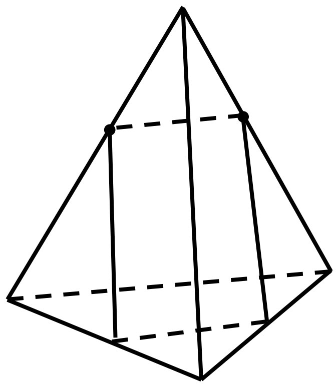

Fig. 20

Solution:

1. We connect points D and E, since the intersection of plane (SAB) with α is the segment DE.
2. Next, we construct the intersections of plane (BSC) with α, which gives segment EN, and of plane (ASC) with α, which gives segment DM, where $EN \parallel SC$ and $DM \parallel SC$.
3. We then connect points M and N, since the intersection of plane (ABC) with α is the segment MN.
4. The quadrilateral $DMNE$ is the required cross-section.
5. We examine triangles $\Delta ASB \sim \Delta DSE$, because they share angle $\measuredangle S$ e, and the ratios $\frac{SA}{SD} = \frac{SB}{SE} = \frac{3}{1}$, so $DE \parallel AB$.
6. Since $AB \parallel DE$ and α passes through DE, it follows that $AB \parallel \alpha$. From this, it follows that $MN \parallel AB$. And since $MN \parallel AB$ and $AB \parallel DE$, we conclude that $MN \parallel DE$.
7. Also, since $DM \parallel SC$ and $EN \parallel SC$, then $DM \parallel EN$.
8. Therefore, the quadrilateral $DMNE$ is a parallelogram.

Problem 5. Construct the section of a regular triangular pyramid $ABCS$ by a plane ρ, which passes through the base edge BC and is perpendicular to the lateral edge SA. Determine the type of the section.

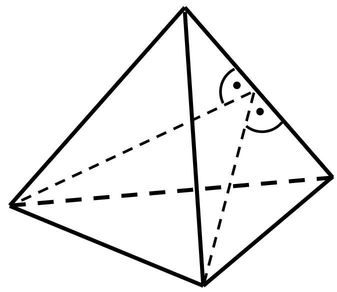

Fig. 21

Solution:

In the regular pyramid $ABCS$, the lateral faces SAB and SAC are congruent isosceles triangles, which implies that $SA = SB = SC$. Therefore, the altitudes from vertices B and C in these triangles intersect at a common point N, which lies on SA, i.e., $N \in SA$. Since $SA \perp NC$ and $SA \perp NB$, it follows that $SA \perp (BNC)$.

Given that $\Delta SAB \cong \Delta SAC$, we also have $BN = CN$. Thus, the triangle ΔBNC is the required cross-section, and it is isosceles.

Successfully solving this problem relies on knowledge of fundamental geometric principles and the ability to apply planar geometry techniques.

One of the cognitive methods that can contribute to the development of spatial imagination and thinking is comparison. It is well known that comparison reveals both the similarities and differences between the objects being studied. This method can help deepen students' reasoning. In this way, the results obtained are more easily understood and remembered.

We will illustrate this by comparing a right circular cylinder and a cone. When studying them, attention should be paid to the fact that:

- Both the cylinder and the cone can be formed by rotating a figure;
- There are elements common to both the cylinder and the cone, but there are also elements unique to each of them;
- In both the cylinder and the cone, a cross-section can be constructed through two generatrices.

After becoming familiar with the elements of the cylinder and the cone, the following task can be given.

Problem 6. Which of the cross-sections passing through two generatrices of a cylinder has the greatest area?

Here, students can easily recognize that the correct answer is the axial (or central) cross-section.

This type of task becomes more engaging when considering the **cone**, specifically:

Problem 7. Which of the cross-sections passing through two generatrices of a cone has the greatest area?

A possible answer here could be: The cross-section with the greatest area will be the axial cross-section, if the angle between the generatrices in the section is right or acute. However, if the angle between the generatrices is obtuse, then the maximum area is achieved in a non-axial cross-section where the angle between the generatrices is right.

Students can be given an assignment to compare the volumes of a cylinder and a cone, if the radii of their bases are R, and their heights are H.

The answer that students will reach is that $V_{cyl} = 3.V_{cone}$. This result does not depend on the particular characteristics of the cylinder and the cone.

In addition to these two problems, the following task could also be given:

Problem 8. Compare the lateral surface areas of the cylinder and the cone, if the radii of their bases are equal to R, and their heights are H.

The answer is that $S_{cone} = S_{cyl}$ if the angle between the generatrices in the axial cross-section of the cone is 120^0. In all other cases, $S_{cone} \neq S_{cyl}$.

Further exploration with students can continue by posing the question: Is it possible for $V_{cyl} = 3.V_{cone}$, given that the radii of the bases of both figures are R, and their heights are H?

Students' reasoning would proceed as follows: Assume that $V_{cyl} = 3.V_{cone}$, i.e., $2\pi RH = 3\pi Rl$. Then, $H = 1{,}5l$, which implies that $H > l$, which is impossible.

In this way, students in their activities can see how comparison facilitates their reasoning and helps them to notice properties of the figures that were previously unnoticed.

Conclusion. By integrating such methods into daily teaching practice, educators can foster students' spatial imagination in a structured and meaningful way. This not only enhances students' understanding of geometric and mathematical concepts but also contributes to the development of broader cognitive skills such as problem-solving, logical reasoning, and abstract thinking. As spatial reasoning is a strong predictor of success in STEM disciplines, its purposeful cultivation equips students with essential tools for navigating complex tasks in science, technology, engineering, and mathematics. In conclusion, prioritizing spatial imagination in contemporary education creates the foundation for developing a generation of learners who not only possess strong mathematical skills, but also demonstrate flexibility, creativity, and the ability to tackle the challenges of a dynamic and technology-driven world.

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Бойкіна Д. Розвиток просторової уяви в математичній освіті.

У цій статті досліджується низка ефективних підходів та стратегій розвитку просторової уяви в учнів у контексті математичної освіти, зокрема з 5 по 12 клас. Просторова уява – здатність подумки візуалізувати та маніпулювати об'єктами та їхніми взаємозв'язками в просторі – є фундаментальним компонентом математичного мислення та вирішення проблем. Вона відіграє життєво важливу роль не лише в геометрії, але й в алгебрі, тригонометрії та реальних застосуваннях, таких як інженерія, архітектура та комп'ютерна графіка.

У статті підкреслюється важливість розвитку просторового мислення на ранніх етапах академічного розвитку учнів та представлено різноманітні методи стимулювання та зміцнення цієї когнітивної навички. До них належать ілюстративні приклади, питання, що спонукають до роздумів, та проблемні завдання, розроблені для заохочення візуального мислення, абстрактного мислення та здатності пов'язувати різні математичні поняття. Особлива увага приділяється проблемам, що передбачають роботу з тривимірними фігурами, дослідження просторових перетворень та інтерпретацію складної візуальної інформації.

Ключові слова: *просторова уява, просторове мислення, просторові навички, вирішення проблем.*

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ШЛЯХИ РОЗВИТКУ КАФЕДРИ ПРИРОДНИЧОГО ПРОФІЛЮ ЯК ВАЖЛИВОЇ КОМПОНЕНТИ МЕДИЧНОЇ ОСВІТИ

Природничі дисципліни відіграють ключову роль у формуванні фахових компетентностей майбутніх лікарів і є невід'ємною складовою системи медичної освіти. Опанування природничих та суміжних наук забезпечує студентів-медиків фундаментальними знаннями про будову і функціонування організму людини у нормі і при патологіях, принципи дії лікарських засобів та основи діагностичних технологій. Саме природничо-наукова підготовка створює базис для свідомого опанування клінічних дисциплін, а напрямки інформаційних технологій забезпечують сучасний рівень цифрової грамотності.

У статті наголошується на важливості постійного розвитку кафедр природничого профілю, адаптації змісту навчання до сучасних досягнень медичної науки, а також на потребі міждисциплінарної інтеграції. Сучасна медична освіта потребує не лише високоякісного викладання базових наук і гнучкості методів навчання, а й активного залучення здобувачів освіти до найрізноманітніших видів діяльності. Такий підхід сприяє підготовці лікарів нового покоління – компетентних, гнучких, орієнтованих на інновації та готових до неперервного професійного розвитку.

Мета дослідження полягає у визначенні можливих шляхів розвитку кафедри природничого профілю у медичному ЗВО у системі медичної освіти на прикладі функціонування кафедри медичної та біологічної фізики і медичної інформатики Буковинського державного медичного університету. Проаналізовано напрямки діяльності кафедри за останні десять років (2014-2024 рр.).

Відзначено основну ідею створеного навчально-методичного комплексу – демонстрація причинно-наслідкових зв'язків у майбутній фаховій діяльності здобувача медичної освіти. Крім цього, акцентовано на важливості розвитку таких видів діяльності, як науково-популярні публікації, створення навчального відеоконтенту, участь здобувачів освіти в роботі наукових гуртків та ін. Підкреслено, важливість проведення тематичних наукових конференцій з природничих наук для висвітлення теоретичних і прикладних досягнень цих наук задля розвитку медицини та створення міждисциплінарних наукових видань.